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AN ASYMPTOTIC THEORY FOR VIBRATING PLATES

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Abstract—We derive the two-dimensional equations of motion for a vibrating plate by means of an asymptotic expansion of the three-dimensional elastic state. The assumptions involved are of mathematical character only and concern the continuity, differentiability and convergence of the series used. The three-dimensional problem is reduced to a two-dimensional eigenvalue problem consisting of a linear fourth order partial differential equation for the deflection of the middle-surface and a proper set of boundary conditions. The eigenvalue appears in the coefficients of the differential equation as well as in the boundary conditions. We discuss the solution of this problem and compute the frequency of stationary plane waves in an infinite plate as an example. The result is compared with the exact solution.

1. INTRODUCTION

Natural frequencies derived by the Kirchhoff theory of plates are affected by errors of order $(h/l)^2$, where $2h$ is the thickness of the plate and l a characteristic wavelength† of the deformation pattern. Although this may be adequate for many applications, the need for a more accurate theory has long been recognized.

The engineering approach to improve the theory is to include certain effects neglected in the Kirchhoff theory, like deformation due to transverse shear, rotational inertia, etc. Notwithstanding the valuable results achieved by many authors using this approach, there is always some uncertainty left with respect to the consistency of these higher order approximations. In fact, no theory derived in this way has taken higher order effects into account in a consistent manner.

In this paper a theory for plates performing small transverse harmonic vibrations is generated by means of an asymptotic expansion of the three-dimensional equations of elasticity in powers of h/l. The principles involved in this procedure are not new but related to early work on plate theory by Poisson and Cauchy and successfully applied by Brod[1] to plates under static surface loads.

We shall tacitly make certain assumptions concerning the differentiability, continuity and convergence of the series used,‡ but on the other hand it must be stressed that our assumptions are solely of mathematical character.

The theory generated in this paper may be carried out to any accuracy desired. The first step yields the lowest order theory which coincides with the Kirchhoff theory. The next non-trivial approximation has inherent errors of orders $(h/l)^4$, etc. The computational work to derive the equations increases rapidly with each step, but otherwise there is no difficulty involved in deriving higher order approximations.

2. BASIC EQUATIONS

Let $x^i = (x, y, z)$ be rectangular coordinates such that $z = \pm h$ are the free surfaces of the plate. Let furthermore the boundary of the plate be defined by the normals to the middle-surface along a simple closed curve C on this surface.

We shall assume that the plate performs small harmonic vibrations of amplitude $u^i = (u, v, w)$. For given conditions at the boundary C the amplitude functions will depend on the

†The term "characteristic wavelength" is admittedly not well defined. We shall see at the end of our work that it can be replaced by a precisely defined length, namely V/ω, where ω is the angular frequency under consideration and V the velocity of sound.

‡It may be recalled that already Saint-Venant questioned the corresponding assumptions in the theories of Poisson and Cauchy (*—elle n'est pas suffisamment fondée, et peut se trouver souvent en défault*). In fact, the region and nature of convergence of these series has hitherto remained unclarified.

thickness of the plate and this dependence will be represented by the following asymptotic expansion:

$$u^i(x, y, z) = \sum_{n=n_0}^{\infty} u_n{}^i(x, y, z)\epsilon^n \tag{2.1}$$

where

$$\epsilon = h/l \tag{2.2}$$

is assumed to be small in comparison with unity.

In the linear theory the amplitude u^i is undertermined and therefore we may without loss of generality put $n_0 = 0$.

The functions $u_n{}^i(x, y, z)$ are now expanded in Taylor series at $z = 0$, i.e.

$$u^i(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} U^i_{n,m}(x, y) z^m \epsilon^n \tag{2.3}$$

where

$$U^i_{n,m}(x, y) = \frac{\partial^m u_n{}^i}{\partial z^m}(x, y, 0) \qquad n, m = 0, 1, \ldots \tag{2.4}$$

are the partial derivatives of $u_n{}^i$ with respect to z at the middle-surface. It follows that the displacements of the middle-surface with which the two-dimensional theory shall deal are given by

$$u^i(x, y, 0) = \sum_{n=0}^{\infty} U^i_{n,0}\epsilon^n. \tag{2.5}$$

The stress tensor $\sigma^{ij}(x, y, z)$ is given in terms of the displacements by

$$\sigma^{ij} = G\left[D^i u^j + D^j u^i + \frac{2\nu}{1-2\nu} g^{ij} D_k u^k\right] \tag{2.6}$$

where G is the shear modulus, ν Poisson's ratio and g^{ij} the metric tensor which in our case of rectangular coordinates reduces to Kronecker's delta. Likewise, the covariant derivative D_i reduces in our case to the partial derivative $D_i = D^i = \partial/\partial x^i$.

After dividing through by the time-factor, the equations of motion take the form

$$D_i \sigma^{ij} + \omega^2 \rho u^j = 0 \tag{2.7}$$

where ω is the angular frequency and ρ the mass density.

Introducing dimensionless stresses $\Sigma^{ij} = \sigma^{ij}/G$ we get

$$\Sigma^{ij} = D^i u^j + D^j u^i + \frac{2\nu}{1-2\nu} g^{ij} D_k u^k. \tag{2.8}$$

From (2.3) and (2.8) it follows that

$$\Sigma^{ij}(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} S^{ij}_{n,m}(x, y) z^m \epsilon^n \tag{2.9}$$

where†

$$S^{\alpha\beta}_{n,m} = D^\alpha U^\beta_{n,m} + D^\beta U^\alpha_{n,m} + \frac{2\nu}{1-2\nu} g^{\alpha\beta}(D_\gamma U^\gamma_{n,m} + W_{n,m+1}) \tag{2.10}$$

$$S^{\alpha 3}_{n,m} = D^\alpha W_{n,m} + U^\alpha_{n,m+1} \tag{2.11}$$

†The tensor indices i, j, k have the range 1, 2, 3 and α, β, γ the range 1, 2. The summation convention applies to i, j, k and α, β, γ. The other indices are not subject to these rules.

$$S^{33}_{n,m} = \frac{2(1-\nu)}{1-2\nu} W_{n,m+1} + \frac{2\nu}{1-2\nu} D_\alpha U^\alpha_{n,m} \tag{2.12}$$

where $W_{n,m} = U^3_{n,m}$. These three equations represent Hooke's law.

The equations of motion (2.7) can now be written in the form

$$D_i \Sigma^{ij} + \Lambda u^j = 0 \tag{2.13}$$

where

$$\Lambda = \rho\omega^2/G. \tag{2.14}$$

The frequency ω and therefore also Λ will depend on the thickness of the plate. Hence Λ too will be represented by an expansion in powers of ϵ.

For any given mode of transverse vibrations the ratio ω/h approaches a finite limit as h tends to zero. For in-plane vibrations on the other hand, ω itself tends to a finite limit as h goes to zero. Depending upon our choice of the power of ϵ (2 or 0) of the lowest order term in the expansion of Λ, the theory will represent transverse or in-plane motion.

In writing

$$\Lambda = \sum_{n=0}^{\infty} \Lambda_n \epsilon^{n+2} \tag{2.15}$$

we deliberately select the transverse vibrations.

By (2.3), (2.9), (2.13), (2.15) and (2.10), (2.11), (2.12) the equations of motion take the form

$$U^\alpha_{n,m+1} + \frac{1}{1-2\nu}(D^\alpha D_\beta U^\beta_{n,m-1} + D^\alpha W_{n,m}) + \Delta U^\alpha_{n,m-1} + \sum_{k=0}^{n-2} \Lambda_k U^\alpha_{n-k-2,m-1} = 0 \tag{2.16}$$

and

$$W_{n,m+1} + \frac{1-2\nu}{2(1-\nu)} \Delta W_{n,m-1} + \frac{1}{2(1-\nu)} D_\alpha U^\alpha_{n,m} + \frac{1-2\nu}{2(1-\nu)} \sum_{k=0}^{n-2} \Lambda_k W_{n-k-2,m-1} = 0 \tag{2.17}$$

where $\Delta = D_\alpha D^\alpha$ is the Laplacian.

The boundary conditions at the free surfaces $z = \pm h$ are given by

$$\Sigma^{i3}(x, y, \pm h) = 0$$

and taking (2.9) and (2.2) into account, this may be written

$$\sum_{m=0}^{n} \frac{(\pm 1)^m}{m!} S^{i3}_{n-m,m} l^m = 0 \qquad n = 0, 1, \ldots \tag{2.18}$$

Adding and subtracting the two equations connected with the plus and minus signs in (2.18) we get the following set of equations for even numbers n:

$$\sum_{m=0}^{n/2} \frac{1}{(2m)!} S^{i3}_{n-2m,2m} l^{2m} = 0 \qquad n = 0, 2, 4, \ldots \tag{2.19}$$

and

$$\sum_{m=0}^{n/2} \frac{1}{(2m+1)!} S^{i3}_{n-2m,2m+1} l^{2m} = 0 \qquad n = 0, 2, 4, \ldots \tag{2.20}$$

The corresponding equations for odd numbers n are similar.

3. PLATE EQUATIONS

In order to derive the two-dimensional plate equations, we must express all "derivatives" $U^i_{n,m}$ $(m > 0)$ in terms of the functions $U^i_{p,0}$ $(p = 0, 1, \dots n)$.

Assuming that the conditions on u^α are homogeneous and independent of u^3 at the boundary C it can be shown that

$$U^\alpha_{n,2m} = W_{n,2m+1} = 0 \qquad n, m = 0, 1, 2, \dots \tag{3.1}$$

and

$$U^\alpha_{2n+1,m} = W_{2n+1,m} = 0 \qquad n, m = 0, 1, 2, \dots \tag{3.2}$$

The first sequence of equations (3.1) shows that the following relations of symmetry hold good:

$$\begin{aligned} u^\alpha(x, y, z) &= -u^\alpha(x, y, -z) \\ u^3(x, y, z) &= u^3(x, y, -z). \end{aligned} \tag{3.3}$$

From the second set of equations (3.2) we may conclude that only the even powers of ϵ contribute to the displacements in (2.1).

By eliminating the remaining "derivatives" $U^i_{n,m}$ with the help of the boundary conditions (2.18), Hooke's law (2.10)–(2.12) and the equations of motion (2.16)–(2.17) we find a sequence of differential relations between the functions $W_{2n,0}$. To achieve this we shall first demonstrate that the derivatives $U^\alpha_{2n,2m+1}$ and $W_{2n,2m}$ can be written in terms of the lower order functions $W_{2p,0}$ $(p = 0, 1, \dots n)$. We prove this by induction assuming that it holds good for $n-1$ and derive the relations for n.

From (2.19) with $i = \alpha$ and (2.11) we get

$$U^\alpha_{2n,1} = -D^\alpha W_{2n,0} - \frac{l^2}{2!}(D^\alpha W_{2n-2,2} + U^\alpha_{2n-2,3}) - \frac{l^{2n}}{(2n)!}(D^\alpha W_{0,2n} + U^\alpha_{0,2n+1}) \tag{3.4}$$

so that the first derivative $U^\alpha_{2n,1}$ can be expressed in terms of $W_{0,0} \dots W_{2n,0}$.

We proceed to the second derivative $W_{2n,2}$. Using (2.20) with $i = 3$ and (2.12) we get

$$\begin{aligned} W_{2n,2} = &-\frac{\nu}{1-\nu} D_\alpha U^\alpha_{2n,1} - \frac{l^2}{3!}\left(W_{2n-2,4} + \frac{\nu}{1-\nu} D_\alpha U^\alpha_{2n-2,3}\right) - \cdots \\ &-\frac{l^{2n}}{(2n+1)!}\left(W_{0,2n+2} + \frac{\nu}{1-\nu} D_\alpha U^\alpha_{0,2n+1}\right). \end{aligned} \tag{3.5}$$

Here all terms on the r.h.s. are given in terms of the functions $W_{0,0} \dots W_{2n,0}$.

With the help of (2.16) and (2.17) the higher order derivatives $U^\alpha_{2n,2m+1}$ and $W_{2n,2m+2}$ $(m = 1, 2, \dots)$ can be determined in terms of $W_{0,0} \dots W_{2n,0}$.

For $n = 0$ we get

$$U^\alpha_{0,2m+1} = (-1)^{m+1}\left(\frac{m}{1-\nu} + 1\right) D^\alpha \Delta^m W_{0,0}$$

and

$$W_{0,2m} = (-1)^{m+1}\left(\frac{m}{1-\nu} - 1\right) \Delta^m W_{0,0}$$

and this concludes the proof by induction.

The second derivative $W_{2n,2}$ can be derived independently of (3.5) from the equations of motion (2.17):

$$W_{2n,2} = -\frac{1-2\nu}{2(1-\nu)} \Delta W_{2n,0} - \frac{1}{2(1-\nu)} D_\alpha U^\alpha_{2n,1} - \frac{1-2\nu}{2(1-\nu)}(\Lambda_0 W_{2n-2,0} + \cdots + \Lambda_{2n-2} W_{0,0}). \tag{3.6}$$

Equating (3.5) and (3.6) yields a differential equation between the functions $W_{0,0}\dots W_{2n,0}$ for each n. Only for $n = 0$ is the equation trivial.

To establish the differential relations we must eliminate the higher order derivatives appearing in the r.h.s. of (3.4) and (3.5) using (2.16) and (2.17). The work is straight forward but tedious and soon grows out of hand. It is therefore helpful and for the higher order equations necessary to apply an automatic procedure for performing the analysis.† The reader is referred to the Appendix for the intermediate formulas, in particular the expressions for the derivatives.

We obtain the following differential equations:

$$\Delta^2 W_{0,0} = \frac{c_1}{l^2}\Lambda_0 W_{0,0}$$

$$\Delta^2 W_{2,0} = \frac{c_1}{l^2}(\Lambda_0 W_{2,0} + \Lambda_2 W_{0,0}) - c_2 l^2 \Delta^3 W_{0,0}$$

$$\Delta^2 W_{4,0} = \frac{c_1}{l^2}(\Lambda_0 W_{4,0} + \Lambda_2 W_{2,0} + \Lambda_4 W_{0,0}) - c_2 l^2 \Delta^3 W_{2,0} - c_3 l^4 \Delta^4 W_{0,0}$$

$$\Delta^2 W_{6,0} = \frac{c_1}{l^2}(\Lambda_0 W_{6,0} + \Lambda_2 W_{4,0} + \Lambda_4 W_{2,0} + \Lambda_6 W_{0,0}) - c_2 l^2 \Delta^3 W_{4,0} - c_3 l^4 \Delta^4 W_{2,0} - c_4 l^6 \Delta^5 W_{0,0} \tag{3.7}$$

etc.

where the first four coefficients turn out to be

$$c_1 = \frac{3(1-\nu)}{2}$$

$$c_2 = \frac{17-7\nu}{15(1-\nu)}$$

$$c_3 = \frac{489-418\nu+62\nu^2}{315(1-\nu)^2}$$

$$c_4 = \frac{11189-14613\nu+4995\nu^2-381\nu^3}{4725(1-\nu)^3}. \tag{3.8}$$

Multiplying the second equation (3.7) by ϵ^2, the third by ϵ^4 etc. and summing up all equations we get by (2.5) and (2.15) one single differential equation for the deflection of the middle-surface,

$$\Delta^2 w = \frac{c_1}{h^2}\Lambda w - c_2 h^2 \Delta^3 w - c_3 h^4 \Delta^4 w - c_4 h^6 \Delta^5 w - \cdots \tag{3.9}$$

where $w(x, y) = u^3(x, y, 0)$.

This equation may be simplified considerably if the higher order Laplacians on the r.h.s. are eliminated using the equation itself. We get

$$\Delta^2 w = \boldsymbol{\lambda}(w - ch^2\Delta w) \tag{3.10}$$

with

$$c = d_2 + d_4 \lambda h^4 + d_6(\lambda h^4)^2 + \cdots \tag{3.11}$$

and

$$\boldsymbol{\lambda} = \lambda/[1 + d_3\lambda h^4 + d_5(\lambda h^4)^2 + \cdots] \tag{3.12}$$

†In the present paper most analytical work has been done using the symbolic manipulation language TENSOR FORMAC implemented on the IBM 370/165 computer of NEUCC (see Jensen and Niordson[2]).

where

$$\lambda = \frac{c_1}{h^2}\Lambda = \frac{3(1-\nu)}{2Gh^2}\omega^2\rho \tag{3.13}$$

$$d_2 = c_2 = \frac{17-7\nu}{15(1-\nu)} \tag{3.14}$$

$$d_3 = c_3 - c_2^2 = \frac{422-424\nu-33\nu^2}{1575(1-\nu)^2} \tag{3.15}$$

$$d_4 = c_4 - 2c_2c_3 + c_2^3 = \frac{7206-10258\nu+2668\nu^2+34\nu^3}{23625(1-\nu)^3} \tag{3.16}$$

etc.

A consistent approximation of order $n > 1$ is obtained by including the coefficients $d_2, d_3, \ldots$ up to and including d_n. The first-order approximation is found when we take $d_2 = d_3 = \cdots = 0$. This approximation coincides with the Kirchhoff theory.

4. MOMENT AND SHEAR FORCE

The stress tensor $\sigma^{ij}(x, y, z)$ is uniquely determined by the functions $U^i_{n,m}$ and thus by the functions $W_{n,0}$. When summing up the contributions to Σ^{ij} according to (2.9) we find that the functions $W_{n,0}$ appear only in the combination $W_{0,0} + \epsilon^2 W_{2,0} + \cdots = w(x, y)$. The stress tensor σ^{ij} can therefore be determined from w. From the stress distribution we find the moment tensor $M^{\alpha\beta}$ and the shear force vector Q^α by

$$M^{\alpha\beta} = -\int_{-h}^{+h} \sigma^{\alpha\beta} z \, dz \tag{4.1}$$

and

$$Q^\alpha = \int_{-h}^{+h} \sigma^{\alpha 3} \, dz. \tag{4.2}$$

Substitution of σ^{ij} yields

$$M^{\alpha\beta} = \frac{4Gh^3}{3}\left(D^\alpha D^\beta \Phi[w] + \frac{\nu}{1-\nu} g^{\alpha\beta}\Psi[w]\right) \tag{4.3}$$

and

$$Q^\alpha = -\frac{4Gh^3}{3(1-\nu)} D^\alpha \chi[w] \tag{4.4}$$

where

$$\Phi[w] = \sum_{p=0}^{\infty} \phi_p h^{2p} \Delta^p w \tag{4.5}$$

$$\Psi[w] = \sum_{p=0}^{\infty} \psi_p h^{2p} \Delta^{p+1} w \tag{4.6}$$

$$\chi[w] = \sum_{p=0}^{\infty} \zeta_p h^{2p} \Delta^{p+1} w \tag{4.7}$$

where

$$\phi_0 = \psi_0 = \zeta_0 = 1 \tag{4.8}$$

applies to the first order (Kirchhoff) theory,

$$\phi_1 = \frac{8+\nu}{10(1-\nu)}$$

$$\psi_1 = \frac{22+3\nu}{30(1-\nu)}$$

$$\zeta_1 = \frac{34-9\nu}{30(1-\nu)} \tag{4.9}$$

applies to the second order theory

$$\phi_2 = \frac{779-152\nu+3\nu^2}{840(1-\nu)^2}$$

$$\psi_2 = \frac{2137-452\nu+9\nu^2}{2520(1-\nu)^2}$$

$$\zeta_2 = \frac{3961-2588\nu+321\nu^2}{2520(1-\nu)^2} \tag{4.10}$$

applies to the third order theory, and

$$\phi_3 = \frac{97132-57465\nu+5808\nu^2+5\nu^3}{75600(1-\nu)^3}$$

$$\psi_3 = \frac{88618-54785\nu+5782\nu^2+5\nu^3}{75600(1-\nu)^3}$$

$$\zeta_3 = \frac{182470-195869\nu+56914\nu^2-3895\nu^3}{75600(1-\nu)^3} \tag{4.11}$$

applies to the fourth order theory, etc.

The higher order Laplacians appearing in $\Phi[w]$, $\Psi[w]$ and $\chi[w]$ may well be reduced to the zero and first order Laplacian by using the differential equation (3.9) or (3.10). Hence the operators Φ, Ψ and χ can be expressed as a linear functions of w and Δw only.

At the boundary C we have the bending moment

$$M_B = M^{\alpha\beta} n_\alpha n_\beta \tag{4.12}$$

and the shear force

$$Q = Q^\alpha n_\alpha \tag{4.13}$$

where n_α is the normal to the boundary curve C.

The bending moment at the boundary is therefore a linear function of $\partial^2 w/\partial n^2$, $\partial^2 \Delta w/\partial n^2$, Δw and w. The derivative $\partial^2 \Delta w/\partial n^2$ may be written $\Delta^2 w - \partial^2 \Delta w/\partial t^2$, or

$$\partial^2 \Delta w/\partial n^2 = \boldsymbol{\lambda}(w - ch^2 \Delta w) - \partial^2 \Delta w/\partial t^2$$

where $\partial/\partial n$ is the normal derivative and $\partial/\partial t$ is the tangential derivative at the boundary.

We see that the eigenvalue problem can be formulated in terms of a fourth-order partial differential equation (3.10) and a proper set of boundary conditions involving the functions w, $\partial w/\partial n$, $\partial^2 w/\partial n^2$ and $\partial^3 w/\partial n^3$.

To solve this eigenvalue problem we may in general have to resort to successive iterations since the eigenvalue appears in the boundary conditions and in the coefficient c of the differential equation. Taking $\boldsymbol{\lambda} = \lambda$ and $\lambda = 0$ in c and the boundary conditions we get a first iteration for λ. This can then be used for getting the next iteration etc.

The numerical value of λh^4 determines the rate of convergence and also the range of applicability of the two-dimensional theory. Clearly λh^4 has to be "small", i.e.

$$\omega h \ll \sqrt{(G/\rho)}$$

for the two-dimensional theory to apply. We can now substitute the somewhat vague term "characteristic length" of the deformation pattern by the more precisely defined length $\sqrt{(G/\rho)}/\omega$.

5. WAVES IN AN INFINITE PLATE

Consider the following solution to the differential eqns (3.9) and (3.10):

$$w = w_0 \sin\frac{\pi x}{a} \tag{5.1}$$

representing the case of a stationary straight-crested flexural wave in a inflnite plate.

Substitution into (3.9) yields the following equation for the frequency:

$$\left(\frac{\pi}{a}\right)^4 = \lambda + c_2h^2\left(\frac{\pi}{a}\right)^6 - c_3h^4\left(\frac{\pi}{a}\right)^8 + c_4h^6\left(\frac{\pi}{a}\right)^{10} - \cdots$$

from which we get

$$\left(\frac{\omega}{\omega_0}\right)^2 = 1 - c_2\left(\frac{\pi h}{a}\right)^2 + c_3\left(\frac{\pi h}{a}\right)^4 - c_4\left(\frac{\pi h}{a}\right)^6 + \cdots \tag{5.2}$$

where

$$\omega_0 = \frac{\pi^2 h}{a^2}\sqrt{\left(\frac{2G}{3(1-\nu)\rho}\right)} \tag{5.3}$$

is the frequency predicted by the Kirchhoff theory. In Fig. 1 the dimensionless frequency ω/ω_0 determined by (5.2) is given as a function of h/a for the first four approximations.

Although eqn (3.10) yields the same result asymptotically as (3.9) from which (5.2) was derived, the truncated solutions will differ for increasing values of h/a. From (3.10) we get, by

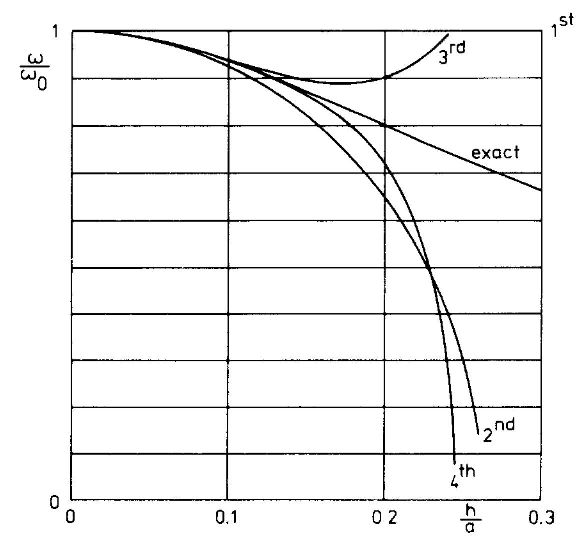

Fig. 1. The frequency of stationary plane waves as a function of the thickness-ratio for the first four approximations according to eqn (5.2) for $\nu = 1/3$. Note that the thickness of the plate is $2h$ and the wavelength $2a$.

substituting (5.1) the following frequency equation:

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{1 + d_3\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{\pi h}{a}\right)^4 + \cdots}{1 + d_2\left(\frac{\pi h}{a}\right)^2 + d_4\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{\pi h}{a}\right)^6 + \cdots}. \tag{5.4}$$

The solution of this equation is shown in Fig. 2. for the first four approximations. Clearly, eqn (3.10) is a better tool for solving this problem than (3.9).

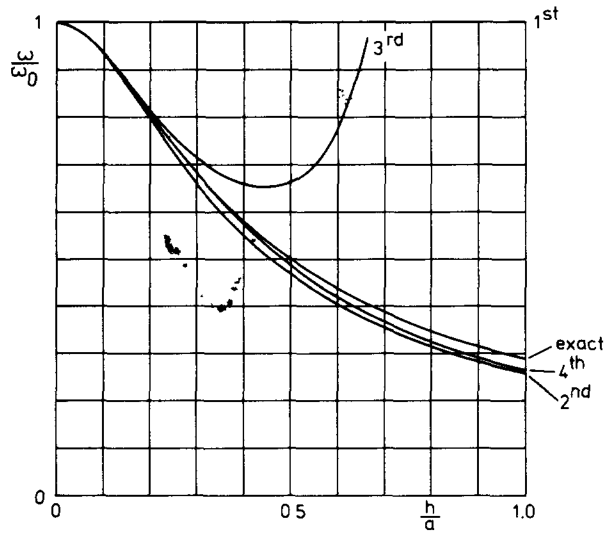

Fig. 2. The frequency of stationary plane waves as a function of the thickness-ratio for the first four approximations according to eqn (5.4) for $\nu = 1/3$. Note that the thickness of the plate is $2h$ and the wavelength $2a$.

The exact solution of the three-dimensional equations of elasticity is known for the case of sinusoidal plane waves in an infinite plate. Lamb[3] has shown that if the motion is symmetrical about the middle-surface, the frequency equation may be written:

$$\frac{4\sqrt{[(1-\alpha\kappa^2)(1-\kappa^2)]}}{(2-\kappa^2)^2} = \frac{\tanh\frac{\pi h}{a}\sqrt{(1-\alpha\kappa^2)}}{\tanh\frac{\pi h}{a}\sqrt{(1-\kappa^2)}} \tag{5.5}$$

where

$$\kappa^2 = \frac{a^2\rho\omega^2}{\pi^2 G} \tag{5.6}$$

and

$$\alpha = \frac{1-2\nu}{2-2\nu}. \tag{5.7}$$

After expanding the hyperbolic functions and simplifying the expression, we can write (5.5) in the form

$$\frac{1-\kappa^2}{\left(1-\frac{1}{2}\kappa^2\right)^2} - \frac{1-\frac{1}{3}\xi^2(1-\alpha\kappa^2)+\frac{2}{15}\xi^4(1-\alpha\kappa^2)^2-\cdots}{1-\frac{1}{3}\xi^2(1-\kappa^2)+\frac{2}{15}\xi^4(1-\kappa^2)^2-\cdots} = 0 \tag{5.8}$$

where

$$\xi = \frac{\pi h}{a}. \tag{5.9}$$

We proceed to write (5.8) with a common denominator and equate the numerator to zero. The common factor κ^2 of the expanded numerator is cancelled and the corresponding solution $\kappa = 0$ is discarded. In the remaining expression we substitute the following power-series expansion for κ^2

$$\kappa^2 = k^2\xi^2(1 - b_2\xi^2 + b_3\xi^4 - b_4\xi^6 + \cdots) \tag{5.10}$$

where the coefficients $k^2, b_2, b_3, \ldots$ are to be determined from the requirement that (5.8) vanishes identically. Equating the coefficients of ξ^{2n} to zero we get the following set of equations:

$$\begin{aligned}
\frac{1}{3}(1-\alpha) - \frac{1}{4}k^2 &= 0 \\
-\frac{4}{15}(1-\alpha) + \left(-\frac{1}{4} + \frac{1}{3}\alpha + \frac{1}{4}b_2\right)k^2 &= 0 \\
\frac{17}{105}(1-\alpha) + \left[\frac{11}{30} - \frac{4}{15}\alpha - \frac{2}{15}\alpha^2 - \frac{1}{12}\alpha k^2 + \left(\frac{1}{4} - \frac{1}{3}\alpha\right)b_2 - \frac{1}{4}b_3\right]k^2 &= 0 \\
-\frac{248}{2835}(1-\alpha) + \left[-\frac{391}{1260} + \frac{17}{105}\alpha(1+\alpha) + \frac{1}{15}\alpha(1+2\alpha)k^2\right. & \\
\left. -\frac{2}{5}k^2 + \left(\frac{4}{15}\alpha + \frac{2}{15}\alpha^2 - \frac{11}{30} + \frac{1}{6}\alpha k^2\right)b_2 + \left(\frac{1}{3}\alpha - \frac{1}{4}\right)b_3 + \frac{1}{4}b_4\right]k^2 &= 0
\end{aligned} \tag{5.11}$$

etc.

The first equation determines k^2, which we substitute in all the other equations. The second equation yields b_2, etc. We find

$$\begin{aligned}
k^2 &= 2/3(1-\nu) \\
b_2 &= (17 - 7\nu)/15(1-\nu) \\
b_3 &= (489 - 418\nu + 62\nu^2)/315(1-\nu)^2 \\
b_4 &= (11189 - 14613\nu + 4995\nu^2 - 381\nu^3)/4725(1-\nu)^3 \\
b_5 &= (602410 - 1059940\nu + 584257\nu^2 - 110090\nu^3 + 5110\nu^4)/155925(1-\nu)^4
\end{aligned} \tag{5.12}$$

etc.

We note that the asymptotic expansion of the exact solution (5.5) due to Lamb is precisely the same as our solution (5.2). This is easily seen when comparing (3.8) with (5.12). The identity is also a check of the correctness of the formulas (3.8). Finally, from (5.12) we are able to determine the coefficient $c_5 = b_5$ for the fifth order theory without the necessity to derive any of the displacement functions $U^i_{n,m}$.

6. DISCUSSION

It is now possible to answer some open questions regarding the accuracy of the engineering theories commonly used for thick plates.

In Mindlin's theory [4], the constant corresponding to c_2 is given as a sum of two terms:

$$c_M = \frac{1}{3} + \frac{2}{3(1-\nu)\kappa^2}.$$

The first represents the influence of rotational inertia and the second one the influence of shear deformation. For κ^2 Mindlin proposes two conflicting values; one being $\pi^2/12$ and the other one depending on ν. For a hypothetical material with Poisson's ratio $\nu = 0.176$ these values coincide, giving $c_M = 1.3170$. For that value of ν however $c_2 = 1.2757$.

In Reissner's theory[5] the constant κ^2 is 5/6 and hence Reissner's value for c_2 is $c_R = (17 - 5\nu)/15(1-\nu)$. Therefore neither Reissner's nor Mindlin's theory gives the correct asymptotic result, but their values are proably close enough for all practical purposes.

It should however be stressed that a second order theory (or for that matter any higher finite order theory) does not necessarily yield better results than Kirchhoff's first order theory. From Fig. 1 we see that eqn (3.9) gives completely misleading results for $h/a > 0.2$, while according to Fig. 2 the even order approximations of (3.10) yield surprisizingly good results also for extremely thick plates.

Of special interest is the conclusion that with the fourth order equation (3.10) there are two boundary conditions to be satisfied as in the classical plate theory rather than the three of Reissner's and Mindlin's theories. For the asymptotic expansion to be valid, the actual stresses at the boundary must coincide with the stresses derived in the theory. In general it is not possible to achieve this. For a free boundary, for example, we may prescribe the bending moment $M_B = 0$ and the effective shear force $Q - \partial M_V/\partial s = 0$ everywhere. Nevertheless there will in general be residual stresses, and the error in neglecting these is difficult to estimate.

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APPENDIX

The following is a listing of the expressions for the derivatives $U^{\alpha}_{n,m}$ and $W_{n,m}$ derived in Section 3. In the list, which is a print-out from the computer, the following simplified notations are used:

$$Unm = U^{\alpha}_{n,m}$$

$$Wnm = W_{n,m} \quad (\text{also} \quad W0A = W_{0,10})$$

$$D^{2n} = \Delta^n$$

$$D^{2n+1} = D^{\alpha}\Delta^n$$

$$V = \nu/(1-\nu)$$

$$LBn = \Lambda_n.$$

In addition it should be noted that in the formulas below the operator D^n operates on the function Wpq appearing in the same term, irrespective of the order in which they stand. Thus, for instance

$$W00\,VD^2 = VD^2\,W00 = \frac{\nu}{1-\nu}\Delta W_{0,0}$$

1. *Approximation*

```
U01 =  - W00 D
--------------

              2
W02 = W00 V D
--------------

          3
U03 = W00 D  ( V + 2 )
----------------------

         4
W04 =  - D   W00 ( 2 V + 1 )
----------------------------
```

$$U21 = - D\ W20 - D^{3}\ W00\ L^{2}\ (V + 1)$$

$$W22 = D^{2}\ V\ W20 + D^{4}\ W00\ L^{2}\ (V + 5/6\ V^{2} + 1/6)$$

$$W22 = 1/2\ W00\ LB0\ (V - 1) + D^{2}\ V\ W20 + D^{4}\ W00\ L^{2}\ (V + 1/2\ V^{2} + 1/2)$$

$$EQU1 = - 3/2\ W00\ LB0 / (V + 1) + D^{4}\ W00\ L^{2}$$

2. *Approximation*

$$U05 = D^{5}\ W00\ (- 2\ V - 3)$$

$$W06 = D^{6}\ W00\ (3\ V + 2)$$

$$U23 = D^{3}\ W20\ (V + 2) + 5\ D^{5}\ W00\ L^{2}\ (2/3\ V + 1/6\ V^{2} + 1/2)$$

$$W24 = - D^{4}\ W20\ (2\ V + 1) + D^{6}\ W00\ L^{2}\ (- 11/3\ V - 2\ V^{2} + 1/3\ V^{3} - 4/3)$$

$$U41 = - W40\ D - D^{3}\ W20\ L^{2}\ (V + 1) - 1/2\ D^{5}\ W00\ L^{4}\ (4\ V + 5/3\ V^{2} + 7/3)$$

$$W42 = W40\ V\ D^{2} + 1/36\ W00\ L^{4}\ D^{6}\ (49\ V + 323/5\ V^{2} + 23\ V^{3} + 37/5) + L^{2}\ D^{4}\ W20\ (V + 5/6\ V^{2} + 1/6)$$

$$W42 = 1/2\ (LB0\ V\ W20 - LB0\ W20 + W00\ V\ LB2 - W00\ LB2) + W40\ V\ D^{2} + 1/12\ W00\ L^{4}\ D^{6}\ (19\ V + 17\ V^{2} + 5\ V^{3} + 7) + L^{2}\ D^{4}\ W20\ (V + 1/2\ V^{2} + 1/2)$$

$$EQU2 = 1/3\ W00\ L^{4}\ D^{6}\ (2\ V + 17/5) + L^{2}\ D^{4}\ W20 + (- 3/2\ LB0\ W20 - 3/2\ W00\ LB2) / (V + 1)$$

3. *Approximation*

$$U07 = W00\ D^{7}\ (3\ V + 4)$$

$$W08 = -W00\,D^{8}\,(4V + 3)$$

$$U25 = W00\,L^{2}\,D^{7}\,(-22/3\,V - 2V^{2} + 1/3\,V^{3} - 5) - D^{5}\,W20\,(2V + 3)$$

$$W26 = W00\,L^{2}\,D^{8}\,(8V + 7/2\,V^{2} - V^{3} + 7/2) + D^{6}\,W20\,(3V + 2)$$

$$U43 = W40\,D^{3}\,(V + 2) + 1/4\,W00\,L^{4}\,D^{7}\,(1277/45\,V + 251/15\,V^{2} + 23/9\,V^{3} + 71/5) + 5\,L^{2}\,D^{5}\,W20\,(2/3\,V + 1/6\,V^{2} + 1/2)$$

$$W44 = -W40\,D^{4}\,(2V + 1) + W00\,L^{4}\,D^{8}\,(-119/18\,V - 63/10\,V^{2} - 101/90\,V^{3} + 1/2\,V^{4} - 29/15) + L^{2}\,D^{6}\,W20\,(-11/3\,V - 2V^{2} + 1/3\,V^{3} - 4/3)$$

$$U61 = -W40\,L^{2}\,D^{3}\,(V + 1) - W60\,D - 1/3\,W00\,L^{6}\,D^{7}\,(34/3\,V + 127/15\,V^{2} + 2V^{3} + 73/15) - L^{4}\,D^{5}\,W20\,(2V + 5/6\,V^{2} + 7/6)$$

$$W62 = W40\,L^{2}\,D^{4}\,(V + 5/6\,V^{2} + 1/6) + W60\,V\,D^{2} + 1/9\,W00\,L^{6}\,D^{8}\,(569/30\,V + 6949/210\,V^{2} + 631/30\,V^{3} + 64/15\,V^{4} + 37/14) + 1/36\,L^{4}\,D^{6}\,W20\,(49V + 323/5\,V^{2} + 23V^{3} + 37/5)$$

$$W62 = -1/2\,W40\,LB0 + 1/2\,W40\,LB0\,V + W40\,L^{2}\,D^{4}\,(V + 1/2\,V^{2} + 1/2) - 1/2\,LB4\,W00 + 1/2\,LB4\,W00\,V + W60\,V\,D^{2} + W00\,L^{6}\,D^{8}\,(27/10\,V + 33/10\,V^{2} + 157/90\,V^{3} + 1/3\,V^{4} + 73/90) + 1/2\,V\,W20\,LB2 - 1/2\,W20\,LB2 + 1/12\,L^{4}\,D^{6}\,W20\,(19V + 17V^{2} + 5V^{3} + 7)$$

$$EQU3 = W40\,L^{2}\,D^{4} + 1/3\,W00\,L^{6}\,D^{8}\,(16/3\,V + 19/15\,V^{2} + 163/35) + 1/3\,L^{4}\,D^{6}\,W20\,(2V + 17/5) + (-3/2\,W40\,LB0 - 3/2\,LB4\,W00 - 3/2\,W20\,LB2)/(V + 1)$$

4. Approximation

$$U09 = W00\, D^{9} \,(-4\,V - 5)$$

$$W0A = W00\, D^{10} \,(5\,V + 4)$$

$$U27 = W00\, L^{2}\, D^{9} \,(13\,V + 7/2\,V^{2} - V^{3} + 17/2) + D^{7}\, W20 \,(3\,V + 4)$$

$$W28 = 2\,W00\, L^{2}\, D^{10} \,(-7\,V - 8/3\,V^{2} + V^{3} - 10/3) - D^{8}\, W20 \,(4\,V + 3)$$

$$U45 = -W40\, D^{5} \,(2\,V + 3) + W00\, L^{4}\, D^{9} \,(-1607/90\,V - 167/15\,V^{2} - 101/90\,V^{3} + 1/2\,V^{4} - 751/90) + L^{2}\, D^{7}\, W20 \,(-22/3\,V - 2\,V^{2} + 1/3\,V^{3} - 5)$$

$$W46 = W40\, D^{6} \,(3\,V + 2) + 1/9\,W00\, L^{4}\, D^{10} \,(635/4\,V + 2633/20\,V^{2} + 221/20\,V^{3} - 29/2\,V^{4} + V^{5} + 1073/20) + L^{2}\, D^{8}\, W20 \,(8\,V + 7/2\,V^{2} - V^{3} + 7/2)$$

$$U63 = 5\,W40\, L^{2}\, D^{5} \,(2/3\,V + 1/6\,V^{2} + 1/2) + W60\, D^{3} \,(V + 2) + 1/3\,W00\, L^{6}\, D^{9} \,(9067/210\,V + 4901/126\,V^{2} + 137/10\,V^{3} + 64/45\,V^{4} + 1159/70) + 1/4\,L^{4}\, D^{7}\, W20 \,(1277/45\,V + 251/15\,V^{2} + 23/9\,V^{3} + 71/5)$$

$$W64 = W40\, L^{2}\, D^{6} \,(-11/3\,V - 2\,V^{2} + 1/3\,V^{3} - 4/3) - W60\, D^{4} \,(2\,V + 1) + 1/18\,W00\, L^{6}\, D^{10} \,(-15433/70\,V - 30353/105\,V^{2} - 4567/35\,V^{3} + 5/2\,V^{4} + 97/10\,V^{5} - 11483/210) + L^{4}\, D^{8}\, W20 \,(-119/18\,V - 63/10\,V^{2} - 101/90\,V^{3} + 1/2\,V^{4} - 29/15)$$

$$U81\;1\; - W40\, L^{4}\, D^{5} \,(2\,V + 5/6\,V^{2} + 7/6) - W60\, L^{2}\, D^{3} \,(V + 1) - W80\, D - 1/9\,W00\, L^{8}\, D^{9} \,(196/3\,V + 5749/84\,V^{2} + 151/5\,V^{3} + 557/120\,V^{4} + 18859/840) - 1/3\,L^{6}\, D^{7}\, W20$$

```
                      2          3
( 34/3 V + 127/15 V    + 2 V    + 73/15 )
-----------------------------------------

                 4    6                      2         3
W82 = 1/36 W40 L   D    ( 49 V + 323/5 V    + 23 V    + 37/5 )
---------------------------------------------------------------
        2   4             2                        2
 + W60 L   D   ( V + 5/6 V    + 1/6 ) + W80 V D    + 1/216 W00
---------------------------------------------------------------
  8   10                                   2                    3
 L   D    ( 267607/350 V + 857599/525 V    + 763547/525 V
---------------------------------------------------------------
                 4            5                             6   8
 + 196127/350 V    + 443/6 V    + 103861/1050 ) + 1/9 L   D
---------------------------------------------------------------
                                2               3            4
W20 ( 569/30 V + 6949/210 V    + 631/30 V    + 64/15 V    + 37/
---------------------------------------------------------------
14 )
----
                                                       4   6
W82 = 1/2 W40 V LB2 - 1/2 W40 LB2 + 1/12 W40 L   D    ( 19 V
---------------------------------------------------------------
     2        3
+ 17 V    + 5 V    + 7 ) + 1/2 LB4 V W20 - 1/2 LB4 W20 - 1/2
---------------------------------------------------------------
                                        2   4                2
W60 LB0 + 1/2 W60 LB0 V + W60 L   D    ( V + 1/2 V    + 1/2 )
---------------------------------------------------------------
                                                2                 8
 - 1/2 LB6 W00 + 1/2 LB6 W00 V + W80 V D    + 1/216 W00 L   D
---------------------------------------------------------------
10                          2               3                 4
     ( 73739/70 V + 11237/7 V    + 41429/35 V    + 4181/10 V
---------------------------------------------------------------
             5                    6   8
 + 557/10 V    + 18859/70 ) + L   D    W20 ( 27/10 V + 33/10 V
---------------------------------------------------------------
2             3          4
  + 157/90 V    + 1/3 V    + 73/90 )
-----------------------------------

                 4    6                           2   4
EQU4 = 1/3 W40 L   D    ( 2 V + 17/5 ) + W60 L   D    + 1/5
---------------------------------------------------------------
     8   10                           2          3
W00 L   D    ( 702/35 V + 3112/315 V    + 34/27 V    + 11189/
---------------------------------------------------------------
               6   8                          2
945 ) + 1/3 L   D    W20 ( 16/3 V + 19/15 V    + 163/35 ) - 3/
---------------------------------------------------------------
2 ( W40 LB2 + LB4 W20 + W60 LB0 + LB6 W00 ) / ( V + 1 )
--------------------------------------------------------
```

Each approximation ends with two expressions for the same derivative $W_{2n,2}$. By equating these two expressions we get one differential equation (3.7) for each n. In successive formulas the eigenvalues $\Lambda_0 \ldots$ are eliminated, using these equations.